Math2050A Term1 2017
Tutorial 2, Sept 21

Exercises

1. Find $\lim_{n\to\infty} \frac{n^2+n}{2n^2-1}$, $\lim_{n\to\infty} \frac{5n^2+2n+3}{n^2+n+2}$, $\lim_{n\to\infty} \frac{n^2+1}{n^3-n^2-1}$. Show your answer by ϵ-$\mathbb{N}$ language.
2. Fix $0 < r < 1$, show that $\lim_{n\to\infty} r^n = 0$.
3. Show that $\lim_{n\to\infty} \sqrt[n]{n} = 1$.
4. Let (a_n) be a sequence in $\mathbb{R}$ such that $\lim_{n\to\infty} a_n = a \in \mathbb{R}$. Let $S_n := a_1 + ... + a_n$. Show that $\lim_{n\to\infty} \frac{S_n}{n} = a$.
5. Let (x_n) be a sequence defined by
$$x_1 = 1, \quad x_{n+1} = \frac{1+x_n}{2+x_n} \quad \forall n \in \mathbb{N}$$
Show that (x_n) converges and find its limit.

Solution

See our textbook[Bartle] p.60 **3.1.11 Examples (b)** for Q2; p.61 **3.1.11 Examples (d)** for Q3;
I forgot to mention **Examples (c)**, which is $\lim_{n\to\infty} \sqrt[n]{c} = 1$ whenever $c > 0$. Please check it yourself.

For Q1, $\lim_{n\to\infty} \frac{n^2+1}{n^3-n^2-1} = 0$, see the following:

Let $\epsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $\frac{6}{N} < \epsilon$, then $\forall n \geq \max\{N, 3\}$, we have (the chosen N and $\max\{N, 3\}$ are due to the computation below)

$$\left|\frac{n^2+1}{n^3-n^2-1} - 0\right| = \frac{n^2+1}{n^3-n^2-1} \leq \frac{n^2+n^2}{n^3-\frac{n^3}{3}-\frac{n^3}{3}} = \frac{6}{n} \leq \frac{6}{N} < \epsilon$$

Supplementary exercises

1. Fix $R > 0$, show (i) $\lim_{n\to\infty} \frac{R^n}{n!} = 0$. Hence, show (ii) $\lim_{n\to\infty} \frac{1}{\sqrt[n]{n!}} = 0$ by ϵ-$\mathbb{N}$ language. (Hint: fixing $R > 0$, by (i), $\exists N \in \mathbb{N}$ such that $\frac{R^n}{n!} < 1$ for all $n \geq N$.)

2. Fix $p \in \mathbb{N}$, $0 < b < 1$. Show $\lim_{n\to\infty} n^p b^n = 0$. The same trick in Q2, 3 works. You may also consult **Ratio test**, **3.2.11 Theorem** in our textbook[Bartle] p.69.

3. Show $\lim_{n\to\infty} \frac{n!}{n^n} = 0$ and $\lim_{n\to\infty} (n!)^{\frac{1}{n^2}} = 1$. You may need $\lim_{n\to\infty} \sqrt[n]{n} = 1$.

4. Consider geometric mean instead. Let (a_n) be a sequence of positive real numbers. Suppose $\lim_{n\to\infty} a_n = a$, show that $\lim_{n\to\infty} \sqrt[n]{a_1 ... a_n} = a$. Check the following:

   Case 1 $(a = 0)$: Let $\epsilon > 0$. There is $N \in \mathbb{N}$ such that $a_n < \epsilon$ for all $n \geq N$. Then, for $n > N$, we have

$$\sqrt[n]{a_1 ... a_n} = \sqrt[n]{a_1 ... a_N}\, \sqrt[n]{a_{N+1} ... a_n} \leq \sqrt[n]{a_1 ... a_N}\, \epsilon^{\frac{n-N}{n}} = \epsilon \sqrt[n]{\frac{a_1 ... a_N}{\epsilon^N}}$$

   Since $\frac{a_1 ... a_N}{\epsilon^N}$ is just a constant > 0, $\lim_{n\to\infty} \sqrt[n]{\frac{a_1 ... a_N}{\epsilon^N}} = 1$. Therefore, there is $N_1 \in \mathbb{N}$ such that $\sqrt[n]{\frac{a_1 ... a_N}{\epsilon^N}} < 2$ for all $n > N_1$.
   To conclude, if $n > \max\{N, N_1\}$, then $\sqrt[n]{a_1 ... a_n} < 2\epsilon$. Case 1 is finished.

   Case 2 $(a \neq 0)$: First, we claim: for any sequence (b_n) of positive real numbers converging to 1 and for any $L > 1$, $\sqrt[n]{b_1 ... b_n} < L$ eventually. (Make sure you know what "eventually" means).

   Proof of claim: There is $N_1 \in \mathbb{N}$ such that $b_n < \frac{L+1}{2}$ for all $n \geq N_1$. There is $N_2 \in \mathbb{N}$ such that $\sqrt[n]{b_1 ... b_{N_1}} < \frac{2L}{L+1}$ for all $n \geq N_2$. Therefore, if $n > \max\{N_1, N_2\}$, then $\sqrt[n]{b_1 ... b_n} < L$.

   Now, fix $L > 1$ and let $b_n := \frac{a_n}{a}$, the claim implies that $\frac{\sqrt[n]{a_1 ... a_n}}{a} < L$ eventually. Let $b_n := \frac{a}{a_n}$, the claim implies that $\frac{1}{L} < \frac{\sqrt[n]{a_1 ... a_n}}{a}$ eventually. Therefore, $\frac{1}{L} < \frac{\sqrt[n]{a_1 ... a_n}}{a} < L$ eventually. This completes the proof.

5. Let $a > 0$, $z_1 > 0$ and $z_{n+1} := \sqrt{a + z_n}$ for all $n \in \mathbb{N}$. Show that the sequence (z_n) is bounded. Show that the sequence converges and find the limit. Similar exercises can be found in our textbook[Bartle] p.77 **Exercises** Q1-7.